Examples of ZetaFunctions Final

```
reset()
```

```
load(DIR + '/ZetaFunctions59Final.sage')
```

```
ZetaFunctions?
```

File: /home/srj/sage-5.9-linux-64bit-ubuntu_10.04.4_lts-x86_64-Linux/local/lib/python2.7/site-packages/sage/all_notebook.py

Type: <type 'type'>

Definition: ZetaFunctions([noargspec])

Docstring:

Class `Zetafunctions` takes a multivariate polynomial as argument for calculate their associated (local) Igusa and Topological zeta functions. This class allows us to get information about: the associated Newton's polyhedron, their faces, the associated cones,...

This class is composed by a multivariate polynomial f of degree n with non-constant term and his associated Newton's polyhedron $\Gamma(f)$.

Methods in ZetaFunctions:

- give_info_facets(self)
- give_info_newton(self, faces = False, cones = False)
- newton_plot(self)
- cones_plot(self)
- give_expected_pole_info(self,d = 1, local = False, weights = None)
- igusa_zeta(self, p = None, dict_Ntau = {}, local = False, weights = None, info = False, check = 'ideals')
- topological_zeta(self, d = 1, local = False, weights = None, info = False, check = 'ideals')
- monodromy_zeta(self, weights = None, char = False, info = False)

Warning

These formulas for the Igusa and Topological zeta functions only work when the given polynomial is NOT DEGENERATED with respect to his associated Newton Polyhedron (see [DenHoo], [DenLoe] and [Var]).

EXAMPLES:

```
sage: R.<x,y,z> = QQ[]
sage: zex = ZetaFunctions(x^2 + y*z)
sage: zex.give_info_newton()
Newton's polyhedron of x^2 + y*z:
    support points = [(2, 0, 0), (0, 1, 1)]
    vertices = [(0, 1, 1), (2, 0, 0)]
    number of proper faces = 13
    Facet 1: x >= 0
    Facet 2: y >= 0
    Facet 3: z >= 0
    Facet 4: x + 2*z - 2 >= 0
    Facet 5: x + 2*y - 2 >= 0
sage: zex.topological_zeta()
(s + 3)/((s + 1)*(2*s + 3))
sage: zex.give_expected_pole_info()
The candidate poles of the (local) topological zeta function (with d =
1) of x^2 + y*z in function of s are:

-3/2 with expected order: 2
The responsible face of maximal dimension is ``tau_0`` = minimal face
who intersects with the diagonal of ambient space:
     tau6: dim 1,  vertices = [(0, 1, 1), (2, 0, 0)],  rays = []
    generators of cone = [(1, 0, 2), (1, 2, 0)], partition into simplicial
    cones = [[(1, 0, 2), (1, 2, 0)]]

-1 with expected order: 1
(If all Vol(tau) are 0, where tau runs through the selected faces that
are no vertices, then the expected order of -1 is 0).
```

REFERENCES:

[DenHoo] J . Denef and K . Hoornaert, "Newton Polyhedra and Igusa's Local Zeta Function.", 2001.

[DenLoe] J . Denef and F . Loeser, "Caracteristiques d'Euler-Poincare;, fonctions zeta locales et modifications analytiques.", 1992.

[HooLoo] K . Hoornaert and D . Loots, "Computer program written in Maple for the calculation of Igusa's local zeta function.", http://www.wis.kuleuven.ac.be/algebra/kathleen.htm, 2000.

[Var] A . N . Varchenko, “Zeta-function of monodromy and Newton’s diagram.”, 1976.

[Viu] J . Viu-Sos, “Funciones zeta y poliedros de Newton: Aspectos teoricos y computacionales.”, 2012.

AUTHORS:

- Kathleen Hoornaert (2000): initial version for Maple
- Juan Viu-Sos (2012): initial version for Sage

Note: All these examples are extracted of the original Maple's work of K. Hoornaert and D. Loots: "Computer program written in Maple for the calculation of Igusa local zeta function", http://www.wis.kuleuven.ac.be/algebra/kathleen.htm, 2000.

Examples for the **Igusa Zeta Function**

Example 1: $x^2 - y^2 + z^3$

```
R.<x,y,z> = QQ[]
zex3 = ZetaFunctions(x^2 - y^2 + z^3)
```

```
type(support_points(x^2 - y^2 + z^3))
```

```
<type 'list'>
```

```
zex3.igusa_zeta?
```

File: /tmp/tmp4oYBk1/<string>

Type: <type ‘instancemethod’>

Definition: zex3.igusa_zeta(p=None, dict_Ntau={}, local=False, weights=None, info=False, check=’ideals’)

Docstring:

Returns the expression of the Igusa zeta function for p a prime number (explicit or abstract), in terms of a symbolic variable `s`.

- `local = True` calculates the local Igusa zeta function (at the origin).
- `weights` – a list $[k_1, \ldots, k_n]$ of weights for the volume form.
- `info = True` gives information of each face τ, the associated cone of τ, and the values `L_tau` and `S_tau` in the process.
- `check` – choose the method to check the non-degeneracy condition (‘default’ or ‘ideals’). If `check = 'no_check'`, degeneracy checking is omitted.

Warning

This formula is only valid when the the given polynomial is NOT DEGENERATED for p with respect to his associated Newton Polyhedron (see [DenHoo]).

In the abstract case `p = None`, you can give a dictionary `dict_Ntau` where:

- The keys are the polynomials f_τ associated of each face τ of the Newton Polyhedron.
- The items are the associated abstract values $N_\tau = \#\{a \in (\mathbb{F}_p - 0)^d \mid f^*_\tau(a) = 0\}$ with $f^*_\tau = \mathbb{F}_p(f_\tau)$, depending of a symbolic variable p.

If the value associated to a face τ_k is not in the dictionary, function introduces a new symbolic variable `N_tauk` to represent N_{τ_k}.

EXAMPLES:

```
sage: R.<x,y,z> = QQ[]; p = var('p')
sage: zex1 = ZetaFunctions(x^2 - y^2 + z^3)
sage: #For p=3 given
sage: zex1.igusa_zeta(p = 3)
2*(3^(2*s + 4) - 3^(s + 1) + 2)*3^(2*s)/((3^(s + 1) - 1)*(3^(3*s + 4) - 1))
sage: #For p arbitrary, we can give the number of solutions over the faces
sage: dNtau1 = { x^2-y^2+z^3 : (p-1)*(p-3), -y^2+z^3 : (p-1)^2, x^2+z^3 : (p-1)^2, x^2-y^2 : 2*(p-1)^2 }
sage: zex1.igusa_zeta(p = None, dict_Ntau = dNtau1)
(p - 1)*(p + p^(2*s + 4) - p^(s + 1) - 1)*p^(2*s)/((p^(s + 1) - 1)*(p^(3*s + 4) - 1))
sage: #
sage: zex2 = ZetaFunctions(x^2 + y*z + z^2)
sage: #For p=3 mod 4, we can give the number of solutions over the faces
sage: dNtau2 = { x^2+y*z+z^2 : (p-1)^2, y*z+z^2 : (p-1)^2,  x^2+y*z : (p-1)^2, x^2+z^2 : 0 }
sage: zex2.igusa_zeta(p = None, dict_Ntau = dNtau2)
(p^(s + 3) - 1)*(p - 1)*p^(2*s)/((p^(s + 1) - 1)*(p^(2*s + 3) - 1))
sage: #For p=1 mod 4
sage: dNtau2bis = { x^2+y*z+z^2 : (p-1)*(p-3), y*z+z^2 : (p-1)^2,  x^2+y*z : (p-1)^2, x^2+z^2 : 2*(p-1)^2 }
sage: zex2.igusa_zeta(p = None, dict_Ntau = dNtau2bis)
(p^(s + 3) - 1)*(p - 1)*p^(2*s)/((p^(s + 1) - 1)*(p^(2*s + 3) - 1))
```

REFERENCES:

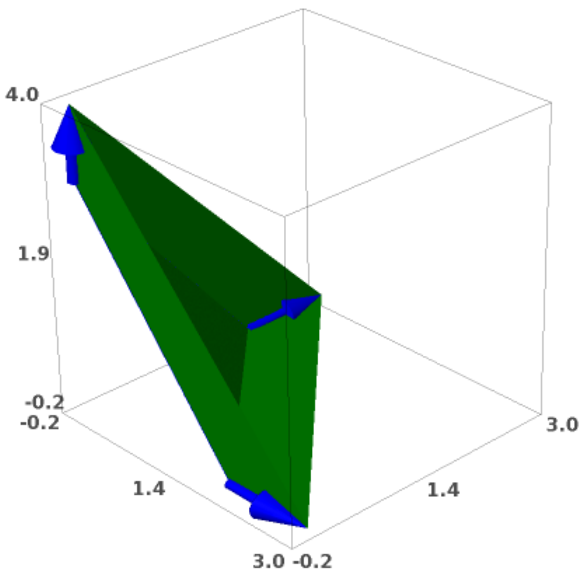
[DenHoo] J . Denef and K . Hoornaert, “Newton Polyhedra and Igusa’s Local Zeta Function.”, 2001.

```
zex3.give_info_newton(faces = True)
```

```
Newton's polyhedron of z^3 + x^2 - y^2:
        support points = [(0, 0, 3), (2, 0, 0), (0, 2, 0)]
        vertices = [(0, 0, 3), (0, 2, 0), (2, 0, 0)]
        number of proper faces = 13
        Facet 1: y >= 0
        Facet 2: z >= 0
        Facet 3: x >= 0
        Facet 4: 3*x + 3*y + 2*z - 6 >= 0
Information about faces:
tau0: dim 0,  vertices = [(0, 0, 3)],  rays = []

tau1: dim 0,  vertices = [(0, 2, 0)],  rays = []

tau2: dim 0,  vertices = [(2, 0, 0)],  rays = []

tau3: dim 1,  vertices = [(0, 0, 3)],  rays = [(0, 0, 1)]

tau4: dim 1,  vertices = [(0, 0, 3), (0, 2, 0)],  rays = []

tau5: dim 1,  vertices = [(0, 2, 0)],  rays = [(0, 1, 0)]

tau6: dim 1,  vertices = [(0, 0, 3), (2, 0, 0)],  rays = []

tau7: dim 1,  vertices = [(0, 2, 0), (2, 0, 0)],  rays = []

tau8: dim 1,  vertices = [(2, 0, 0)],  rays = [(1, 0, 0)]

tau9: dim 2,  vertices = [(0, 0, 3), (2, 0, 0)],  rays = [(0, 0, 1),
(1, 0, 0)]

tau10: dim 2,  vertices = [(0, 2, 0), (2, 0, 0)],  rays = [(0, 1,
0), (1, 0, 0)]

tau11: dim 2,  vertices = [(0, 0, 3), (0, 2, 0)],  rays = [(0, 0,
1), (0, 1, 0)]

tau12: dim 2,  vertices = [(0, 0, 3), (0, 2, 0), (2, 0, 0)],  rays =
[]
```

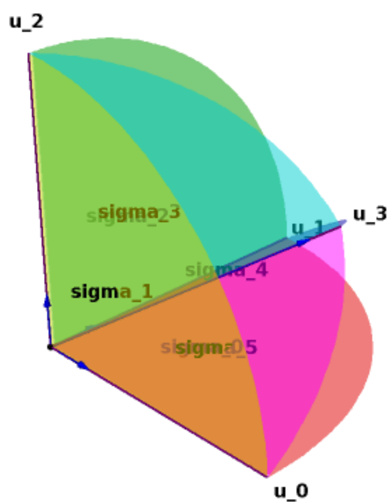
```
zex3.newton_plot()
```

Sleeping... Make Interactive




```
zex3.cones_plot()
```

Sleeping... Make Interactive



$p = 3$:

```
zex3.igusa_zeta(p=3)
```

```
2*(3^(2*s + 4) - 3^(s + 1) + 2)*3^(2*s)/((3^(s + 1) - 1)*(3^(3*s +
4) - 1))
```

For p arbitrary, without information over the faces:

```
zex3.igusa_zeta()
```

```
(N_Gamma - N_tau10 - N_tau11 - N_tau12 + N_tau4 + N_tau6 + N_tau7 -
N_tau9 + p^(7*s + 12) - p^(7*s + 11) - p^(7*s + 9) + p^(7*s + 8) -
p^(6*s + 11) + p^(6*s + 10) + p^(6*s + 8) - p^(6*s + 7) + p^(5*s +
```

```
9) - p^(5*s + 8) - p^(5*s + 7) + p^(5*s + 6) - p^(4*s + 8) +
2*p^(4*s + 7) - p^(4*s + 6) + p^(3*s + 5) - 2*p^(3*s + 4) + p^(3*s +
3) - p^(2*s + 5) + p^(2*s + 4) + p^(2*s + 3) - p^(2*s + 2) -
p^s*N_Gamma + p^s*N_tau10 + p^s*N_tau11 + p^s*N_tau12 - p^s*N_tau4 -
p^s*N_tau6 - p^s*N_tau7 + p^s*N_tau9 - N_Gamma*p - N_Gamma*p^(7*s +
9) + N_Gamma*p^(7*s + 8) + N_Gamma*p^(6*s + 9) - N_Gamma*p^(6*s + 8)
+ N_Gamma*p^(s + 1) - N_tau10*p^(7*s + 8) + N_tau10*p^(6*s + 8) -
N_tau11*p^(7*s + 8) + N_tau11*p^(6*s + 8) + N_tau12*p - N_tau12*p^(s
+ 1) - N_tau7*p^(5*s + 6) + N_tau7*p^(4*s + 6) - N_tau7*p^(3*s + 3)
+ N_tau7*p^(2*s + 3) - N_tau9*p^(7*s + 8) + N_tau9*p^(6*s +
8))/((p^(s + 1) - 1)*(p^(3*s + 4) - 1)*(p^(3*s + 4) + 1)*(p -
1)*p^2)
```

For p arbitrary, with the number of solutions over the faces:

```
dNtau3 = { x^2-y^2+z^3 : (p-1)*(p-3), -y^2+z^3 : (p-1)^2, x^2+z^3 : (p-1)^2, x^2-y^2 : 2*(p-1)^2 }
```

```
zex3.igusa_zeta(dict_Ntau = dNtau3)
```

```
(p - 1)*(p + p^(2*s + 4) - p^(s + 1) - 1)*p^(2*s)/((p^(s + 1) -
1)*(p^(3*s + 4) - 1))
```

Example 4: $(x-y)^2+z$

```
zex4 = ZetaFunctions((x - y)^2 + z)
```

$p = 7$:

```
zex4.igusa_zeta(p=7)
```

```
The formula for Igusa Zeta function is not valid:
The polynomial is degenerated at least with respect to the face tau
= {dim 1,  vertices = [(0, 2, 0), (2, 0, 0)],  rays = []} over
GF(7)!
NaN
```

For an arbitrary p:

```
zex4.igusa_zeta()
```

```
The formula for Igusa Zeta function is not valid:
The polynomial is degenerated at least with respect to the face tau
= {dim 1,  vertices = [(0, 2, 0), (2, 0, 0)],  rays = []} over the
complex numbers!
NaN
```

Example 5: x^2+yz+z^2

```
zex5 = ZetaFunctions(x^2 + y*z + z^2)
```

For $p = 3 \bmod 4$, we can give the number of solutions over the faces:

```
dNtau5 = { x^2+y*z+z^2 : (p-1)^2, y*z+z^2 : (p-1)^2,  x^2+y*z : (p-1)^2, x^2+z^2 : 0 }
zex5.igusa_zeta(dict_Ntau = dNtau5, info = True)
```

```
Gamma: total polyhedron
L_gamma = -((p - 1)^2*(p^s - 1)*p/(p^(s + 1) - 1) - (p - 1)^3)/p^3


tau0: dim 0,  vertices = [(0, 0, 2)],  rays = []
generators of cone = [(0, 1, 0), (1, 0, 0), (1, 1, 1)], partition
into simplicial cones = [[(0, 1, 0), (1, 0, 0), (1, 1, 1)]]
multiplicities = [1], integral points = [[(0, 0, 0)]]
N_tau = 0, L_tau = (p - 1)^3/p^3 , S_tau = 1/((p^(2*s + 3) - 1)*(p -
```

```
1)^2)


tau1: dim 0,  vertices = [(0, 1, 1)],  rays = []
generators of cone = [(1, 0, 0), (1, 0, 2), (1, 1, 1)], partition
into simplicial cones = [[(1, 0, 0), (1, 0, 2), (1, 1, 1)]]
multiplicities = [2], integral points = [[(0, 0, 0), (1, 0, 1)]]
N_tau = 0, L_tau = (p - 1)^3/p^3 , S_tau = (p^(s + 2) + 1)/((p^(2*s
+ 3) - 1)^2*(p - 1))


tau2: dim 0,  vertices = [(2, 0, 0)],  rays = []
generators of cone = [(0, 1, 0), (0, 0, 1), (1, 0, 2), (1, 1, 1)],
partition into simplicial cones = [[(1, 0, 2), (0, 1, 0)], [(1, 0,
2), (0, 0, 1), (0, 1, 0)], [(1, 0, 2), (1, 1, 1), (0, 1, 0)]]
multiplicities = [1, 1, 1], integral points = [[(0, 0, 0)], [(0, 0,
0)], [(0, 0, 0)]]
N_tau = 0, L_tau = (p - 1)^3/p^3 , S_tau = 1/((p^(2*s + 3) - 1)*(p -
1)) + 1/((p^(2*s + 3) - 1)*(p - 1)^2) + 1/((p^(2*s + 3) - 1)^2*(p -
1))


tau3: dim 1,  vertices = [(0, 0, 2)],  rays = [(0, 0, 1)]
generators of cone = [(0, 1, 0), (1, 0, 0)], partition into
simplicial cones = [[(0, 1, 0), (1, 0, 0)]]
multiplicities = [1], integral points = [[(0, 0, 0)]]
N_tau = 0, L_tau = (p - 1)^3/p^3 , S_tau = (p - 1)^(-2)


tau4: dim 1,  vertices = [(0, 0, 2), (0, 1, 1)],  rays = []
generators of cone = [(1, 0, 0), (1, 1, 1)], partition into
simplicial cones = [[(1, 0, 0), (1, 1, 1)]]
multiplicities = [1], integral points = [[(0, 0, 0)]]
N_tau = (p - 1)^2, L_tau = (p - 1)^2*(p^(s + 2) - 2*p^(s + 1) +
1)/((p^(s + 1) - 1)*p^3) , S_tau = 1/((p^(2*s + 3) - 1)*(p - 1))


tau5: dim 1,  vertices = [(0, 1, 1)],  rays = [(0, 1, 0)]
generators of cone = [(1, 0, 0), (1, 0, 2)], partition into
simplicial cones = [[(1, 0, 0), (1, 0, 2)]]
multiplicities = [2], integral points = [[(0, 0, 0), (1, 0, 1)]]
N_tau = 0, L_tau = (p - 1)^3/p^3 , S_tau = (p^(s + 2) + 1)/((p^(2*s
+ 3) - 1)*(p - 1))


tau6: dim 1,  vertices = [(0, 0, 2), (2, 0, 0)],  rays = []
generators of cone = [(0, 1, 0), (1, 1, 1)], partition into
simplicial cones = [[(0, 1, 0), (1, 1, 1)]]
multiplicities = [1], integral points = [[(0, 0, 0)]]
N_tau = 0, L_tau = (p - 1)^3/p^3 , S_tau = 1/((p^(2*s + 3) - 1)*(p -
1))


tau7: dim 1,  vertices = [(2, 0, 0)],  rays = [(0, 1, 0)]
generators of cone = [(0, 0, 1), (1, 0, 2)], partition into
simplicial cones = [[(0, 0, 1), (1, 0, 2)]]
multiplicities = [1], integral points = [[(0, 0, 0)]]
N_tau = 0, L_tau = (p - 1)^3/p^3 , S_tau = 1/((p^(2*s + 3) - 1)*(p -
1))


tau8: dim 1,  vertices = [(0, 1, 1), (2, 0, 0)],  rays = []
generators of cone = [(1, 0, 2), (1, 1, 1)], partition into
simplicial cones = [[(1, 0, 2), (1, 1, 1)]]
multiplicities = [1], integral points = [[(0, 0, 0)]]
N_tau = (p - 1)^2, L_tau = (p - 1)^2*(p^(s + 2) - 2*p^(s + 1) +
1)/((p^(s + 1) - 1)*p^3) , S_tau = (p^(2*s + 3) - 1)^(-2)
```

```
tau9: dim 1,  vertices = [(2, 0, 0)],  rays = [(1, 0, 0)]
generators of cone = [(0, 1, 0), (0, 0, 1)], partition into
simplicial cones = [[(0, 1, 0), (0, 0, 1)]]
multiplicities = [1], integral points = [[(0, 0, 0)]]
N_tau = 0, L_tau = (p - 1)^3/p^3 , S_tau = (p - 1)^(-2)


tau10: dim 2,  vertices = [(0, 0, 2), (2, 0, 0)],  rays = [(0, 0,
1), (1, 0, 0)]
generators of cone = [(0, 1, 0)], partition into simplicial cones =
[[(0, 1, 0)]]
multiplicities = [1], integral points = [[(0, 0, 0)]]
N_tau = 0, L_tau = (p - 1)^3/p^3 , S_tau = 1/(p - 1)


tau11: dim 2,  vertices = [(0, 0, 2), (0, 1, 1)],  rays = [(0, 0,
1), (0, 1, 0)]
generators of cone = [(1, 0, 0)], partition into simplicial cones =
[[(1, 0, 0)]]
multiplicities = [1], integral points = [[(0, 0, 0)]]
N_tau = (p - 1)^2, L_tau = (p - 1)^2*(p^(s + 2) - 2*p^(s + 1) +
1)/((p^(s + 1) - 1)*p^3) , S_tau = 1/(p - 1)


tau12: dim 2,  vertices = [(2, 0, 0)],  rays = [(0, 1, 0), (1, 0,
0)]
generators of cone = [(0, 0, 1)], partition into simplicial cones =
[[(0, 0, 1)]]
multiplicities = [1], integral points = [[(0, 0, 0)]]
N_tau = 0, L_tau = (p - 1)^3/p^3 , S_tau = 1/(p - 1)


tau13: dim 2,  vertices = [(0, 1, 1), (2, 0, 0)],  rays = [(0, 1,
0)]
generators of cone = [(1, 0, 2)], partition into simplicial cones =
[[(1, 0, 2)]]
multiplicities = [1], integral points = [[(0, 0, 0)]]
N_tau = (p - 1)^2, L_tau = (p - 1)^2*(p^(s + 2) - 2*p^(s + 1) +
1)/((p^(s + 1) - 1)*p^3) , S_tau = 1/(p^(2*s + 3) - 1)


tau14: dim 2,  vertices = [(0, 0, 2), (0, 1, 1), (2, 0, 0)],  rays =
[]
generators of cone = [(1, 1, 1)], partition into simplicial cones =
[[(1, 1, 1)]]
multiplicities = [1], integral points = [[(0, 0, 0)]]
N_tau = (p - 1)^2, L_tau = (p - 1)^2*(p^(s + 2) - 2*p^(s + 1) +
1)/((p^(s + 1) - 1)*p^3) , S_tau = 1/(p^(2*s + 3) - 1)


(p^(s + 3) - 1)*(p - 1)*p^(2*s)/((p^(s + 1) - 1)*(p^(2*s + 3) - 1))
```

For $p = 1 \bmod 4$:

```
dNtau5bis = { x^2+y*z+z^2 : (p-1)*(p-3), y*z+z^2 : (p-1)^2,  x^2+y*z : (p-1)^2, x^2+z^2 : 2*(p-1)^2 }
zex5.igusa_zeta(dict_Ntau = dNtau5bis)
```

```
(p^(s + 3) - 1)*(p - 1)*p^(2*s)/((p^(s + 1) - 1)*(p^(2*s + 3) - 1))
```

Example 6: $x^2z + y^2z + u^3$

```
S.<x,y,z,u> = ZZ[]
zex6 = ZetaFunctions(x^2*z + y^2*z + u^3)
```

For $p = 1 \bmod 4$, with the number of solutions over the faces:

```
dNtau6 = { x^2*z+y^2*z+u^3 : (p-1)^2*(p-3), x^2*z+u^3 : (p-1)^3, y^2*z + u^3: (p-1)^3,  x^2*z+y^2*z
: 2*(p-1)^3}
zex6.igusa_zeta(dict_Ntau = dNtau6)
```

```
(p - 1)*(p^(4*s + 8) - 3*p^(3*s + 5) + 2*p^(3*s + 4) + 3*p^(2*s + 5)
- 6*p^(2*s + 4) + 3*p^(2*s + 3) + 2*p^(s + 4) - 3*p^(s + 3) +
1)*p^(3*s)/((p^(s + 1) - 1)*(p^(3*s + 4) - 1)^2)
```

Local for $p = 1 \bmod 4$, with the number of solutions over the faces:

```
zex6.igusa_zeta(local = True, dict_Ntau = dNtau6)
```

```
(p - 1)*(p^(4*s + 8) - 3*p^(3*s + 5) + 2*p^(3*s + 4) + 3*p^(2*s + 5)
- 6*p^(2*s + 4) + 3*p^(2*s + 3) + 2*p^(s + 4) - 3*p^(s + 3) +
1)/((p^(s + 1) - 1)*(p^(3*s + 4) - 1)^2*p^4)
```

Local for $p = 3 \bmod 4$, with the number of solutions over the faces::

```
dNtau6bis = { x^2*z+y^2*z+u^3 : (p-1)^3, x^2*z+u^3 : (p-1)^3, y^2*z + u^3: (p-1)^3,  x^2*z+y^2*z : 0}
zex6.igusa_zeta(local = True, dict_Ntau = dNtau6bis, info = True)
```

```
tau0: dim 0,  vertices = [(0, 0, 0, 3)],  rays = []
generators of cone = [(0, 1, 0, 0), (0, 0, 3, 1), (1, 0, 0, 0), (0,
0, 1, 0), (3, 3, 0, 2)], partition into simplicial cones = [[(1, 0,
0, 0), (0, 0, 3, 1), (0, 1, 0, 0)], [(0, 1, 0, 0), (1, 0, 0, 0), (0,
0, 3, 1), (0, 0, 1, 0)], [(3, 3, 0, 2), (1, 0, 0, 0), (0, 0, 3, 1),
(0, 1, 0, 0)]]
multiplicities = [1, 1, 6], integral points = [[(0, 0, 0, 0)], [(0,
0, 0, 0)], [(0, 0, 0, 0), (3, 3, 1, 2), (2, 2, 2, 2), (2, 2, 0, 1),
(1, 1, 1, 1), (1, 1, 2, 1)]]
N_tau = 0, L_tau = (p - 1)^4/p^4 , S_tau = (p^(6*s + 9) + p^(6*s +
8) + 2*p^(3*s + 5) + p^(3*s + 4) + 1)/((p^(3*s + 4) - 1)*(p^(6*s +
8) - 1)*(p - 1)^2) + 1/((p^(3*s + 4) - 1)*(p - 1)^2) + 1/((p^(3*s +
4) - 1)*(p - 1)^3)


tau1: dim 0,  vertices = [(0, 2, 1, 0)],  rays = []
generators of cone = [(0, 0, 0, 1), (0, 0, 3, 1), (1, 0, 0, 0), (3,
3, 0, 2)], partition into simplicial cones = [[(0, 0, 0, 1), (0, 0,
3, 1), (1, 0, 0, 0), (3, 3, 0, 2)]]
multiplicities = [9], integral points = [[(0, 0, 0, 0), (1, 1, 0,
1), (2, 2, 0, 2), (0, 0, 1, 1), (1, 1, 1, 1), (2, 2, 1, 2), (0, 0,
2, 1), (1, 1, 2, 2), (2, 2, 2, 2)]]
N_tau = 0, L_tau = (p - 1)^4/p^4 , S_tau = (p^(6*s + 8) + p^(5*s +
7) + 2*p^(4*s + 6) + p^(3*s + 4) + 2*p^(2*s + 3) + p^(s + 2) +
1)/((p^(3*s + 4) - 1)*(p^(6*s + 8) - 1)*(p - 1)^2)


tau2: dim 0,  vertices = [(2, 0, 1, 0)],  rays = []
generators of cone = [(0, 1, 0, 0), (0, 0, 0, 1), (0, 0, 3, 1), (3,
3, 0, 2)], partition into simplicial cones = [[(0, 1, 0, 0), (0, 0,
0, 1), (0, 0, 3, 1), (3, 3, 0, 2)]]
multiplicities = [9], integral points = [[(0, 0, 0, 0), (1, 1, 0,
1), (2, 2, 0, 2), (0, 0, 1, 1), (1, 1, 1, 1), (2, 2, 1, 2), (0, 0,
2, 1), (1, 1, 2, 2), (2, 2, 2, 2)]]
N_tau = 0, L_tau = (p - 1)^4/p^4 , S_tau = (p^(6*s + 8) + p^(5*s +
7) + 2*p^(4*s + 6) + p^(3*s + 4) + 2*p^(2*s + 3) + p^(s + 2) +
1)/((p^(3*s + 4) - 1)*(p^(6*s + 8) - 1)*(p - 1)^2)


tau11: dim 1,  vertices = [(0, 0, 0, 3), (0, 2, 1, 0)],  rays = []
generators of cone = [(0, 0, 3, 1), (1, 0, 0, 0), (3, 3, 0, 2)],
partition into simplicial cones = [[(0, 0, 3, 1), (1, 0, 0, 0), (3,
3, 0, 2)]]
multiplicities = [3], integral points = [[(0, 0, 0, 0), (1, 1, 1,
1), (2, 2, 2, 2)]]
N_tau = (p - 1)^3, L_tau = (p - 1)^3*(p^(s + 2) - 2*p^(s + 1) +
1)/((p^(s + 1) - 1)*p^4) , S_tau = (p^(6*s + 8) + p^(3*s + 4) +
1)/((p^(3*s + 4) - 1)*(p^(6*s + 8) - 1)*(p - 1))
```

```
tau17: dim 1,  vertices = [(0, 0, 0, 3), (2, 0, 1, 0)],  rays = []
generators of cone = [(0, 1, 0, 0), (0, 0, 3, 1), (3, 3, 0, 2)],
partition into simplicial cones = [[(0, 1, 0, 0), (0, 0, 3, 1), (3,
3, 0, 2)]]
multiplicities = [3], integral points = [[(0, 0, 0, 0), (1, 1, 1,
1), (2, 2, 2, 2)]]
N_tau = (p - 1)^3, L_tau = (p - 1)^3*(p^(s + 2) - 2*p^(s + 1) +
1)/((p^(s + 1) - 1)*p^4) , S_tau = (p^(6*s + 8) + p^(3*s + 4) +
1)/((p^(3*s + 4) - 1)*(p^(6*s + 8) - 1)*(p - 1))


tau20: dim 1,  vertices = [(0, 2, 1, 0), (2, 0, 1, 0)],  rays = []
generators of cone = [(0, 0, 0, 1), (0, 0, 3, 1), (3, 3, 0, 2)],
partition into simplicial cones = [[(0, 0, 0, 1), (0, 0, 3, 1), (3,
3, 0, 2)]]
multiplicities = [9], integral points = [[(0, 0, 0, 0), (0, 0, 1,
1), (0, 0, 2, 1), (1, 1, 0, 1), (1, 1, 1, 1), (1, 1, 2, 2), (2, 2,
0, 2), (2, 2, 1, 2), (2, 2, 2, 2)]]
N_tau = 0, L_tau = (p - 1)^4/p^4 , S_tau = (p^(6*s + 8) + p^(5*s +
7) + 2*p^(4*s + 6) + p^(3*s + 4) + 2*p^(2*s + 3) + p^(s + 2) +
1)/((p^(3*s + 4) - 1)*(p^(6*s + 8) - 1)*(p - 1))


tau22: dim 2,  vertices = [(0, 0, 0, 3), (0, 2, 1, 0), (2, 0, 1,
0)],  rays = []
generators of cone = [(0, 0, 3, 1), (3, 3, 0, 2)], partition into
simplicial cones = [[(0, 0, 3, 1), (3, 3, 0, 2)]]
multiplicities = [3], integral points = [[(0, 0, 0, 0), (1, 1, 1,
1), (2, 2, 2, 2)]]
N_tau = (p - 1)^3, L_tau = (p - 1)^3*(p^(s + 2) - 2*p^(s + 1) +
1)/((p^(s + 1) - 1)*p^4) , S_tau = (p^(6*s + 8) + p^(3*s + 4) +
1)/((p^(3*s + 4) - 1)*(p^(6*s + 8) - 1))


(p^(s + 2) + 1)*(p - 1)*(p^(3*s + 6) - p^(2*s + 4) - p^(2*s + 3) +
p^(s + 3) + p^(s + 2) - 1)/((p^(s + 1) - 1)*(p^(3*s + 4) -
1)*(p^(3*s + 4) + 1)*p^4)
```

Examples for the **Topological Zeta Function**

Example 10: $x^2 + yz$

```
R.<x,y,z> = QQ[]
zex10 = ZetaFunctions(R(x^2 + y*z))
```

```
zex10.topological_zeta?
```

File: /tmp/tmpGa4Nuv/<string>

Type: <type 'instancemethod'>

Definition: zex10.topological_zeta(d=1, local=False, weights=None, info=False, check='ideals')

Docstring:

Returns the expression of the Topological zeta function $Z_{top,f}^{(d)}$ for $d \geq 1$, in terms of the symbolic variable s:

- `local = True` calculates the local Topological zeta function (at the origin).
- `weights` – a list $[k_1, \ldots, k_n]$ of weights for the volume form.
- d – (default:1) an integer. We consider only the divisor whose multiplicity is a multiple of d (see [DenLoe]).
- `info = True` gives information of each face τ, the associated cone of τ, and the values J_τ and $dim(\tau)! \cdot Vol(\tau)$ in the process (see [DenLoe]).
- `check` – choose the method to check the non-degeneracy condition ('default' or 'ideals'). If `check = 'no_check'`, degeneracy checking is omitted.

Warning

This formula is only valid when the the given polynomial is NOT DEGENERATED with respect to his associated Newton Polyhedron (see [DenLoe]).

EXAMPLES:

```
sage: R.<x,y,z> = QQ[]
sage: zex1 = ZetaFunctions(x^2 + y*z)
sage: zex1.topological_zeta()
(s + 3)/((s + 1)*(2*s + 3))
sage: #For d = 2
sage: zex1.topological_zeta(d = 2)
1/(2*s + 3)
```

REFERENCES:

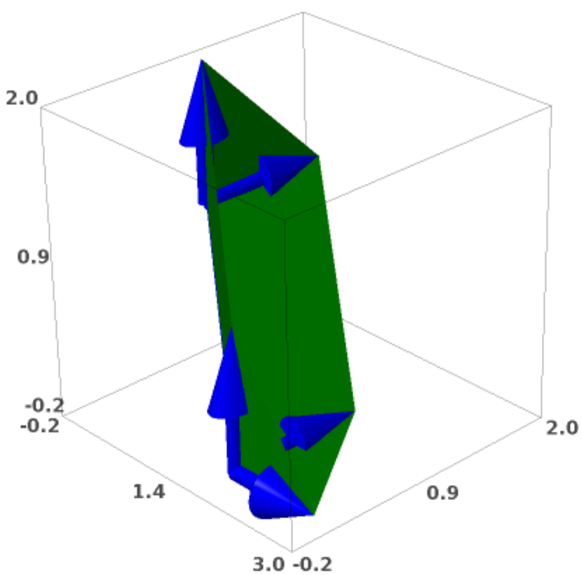
[DenLoe] (1, 2, 3) J . Denef and F . Loeser, “Caracteristiques d’Euler-Poincare;, fonctions zeta locales et modifications analytiques.”, 1992.


```
zex10.give_info_newton()
```

```
Newton's polyhedron of x^2 + y*z:
        support points = [(2, 0, 0), (0, 1, 1)]
        vertices = [(0, 1, 1), (2, 0, 0)]
        number of proper faces = 13
        Facet 1: x >= 0
        Facet 2: y >= 0
        Facet 3: z >= 0
        Facet 4: x + 2*z - 2 >= 0
        Facet 5: x + 2*y - 2 >= 0
```

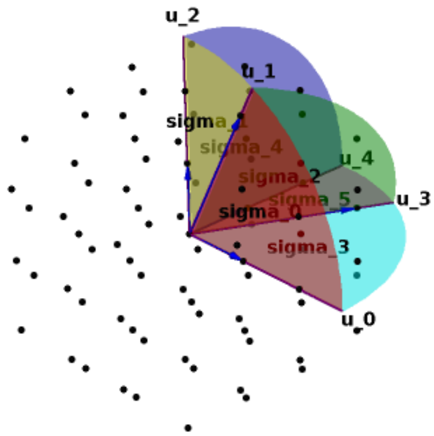
```
zex10.newton_plot()
```

Sleeping... Make Interactive




```
zex10.cones_plot()
```

Sleeping... Make Interactive




```
zex10.give_expected_pole_info()
```

```
The candidate poles of the (local) topological zeta function (with d
= 1) of x^2 + y*z in function of s are:

-3/2 with expected order: 2
The responsible face of maximal dimension is ``tau_0`` = minimal
face who intersects with the diagonal of ambient space:
         tau6: dim 1,  vertices = [(0, 1, 1), (2, 0, 0)],  rays = []
        generators of cone = [(1, 0, 2), (1, 2, 0)], partition into
simplicial cones = [[(1, 0, 2), (1, 2, 0)]]

-1 with expected order: 1
(If all Vol(tau) are 0, where tau runs through the selected faces
that are no vertices, then the expected order of -1 is 0).
```

```
zex10.topological_zeta(info = True)
```

```
Gamma: total polyhedron
J_gamma = 1 , dim_Gamma!*Vol(Gamma) = 0


tau0: dim 0,  vertices = [(0, 1, 1)],  rays = []
generators of cone = [(1, 0, 0), (1, 0, 2), (1, 2, 0)], partition
into simplicial cones = [[(1, 0, 0), (1, 0, 2), (1, 2, 0)]]
multiplicities = [4], integral points = [[(0, 0, 0), (1, 1, 0), (1,
0, 1), (1, 1, 1)]]
J_tau = 4/(2*s + 3)^2 , dim_tau!*Vol(tau) = 1


tau1: dim 0,  vertices = [(2, 0, 0)],  rays = []
generators of cone = [(0, 1, 0), (0, 0, 1), (1, 0, 2), (1, 2, 0)],
partition into simplicial cones = [[(1, 0, 2), (0, 1, 0)], [(1, 0,
2), (0, 0, 1), (0, 1, 0)], [(1, 0, 2), (1, 2, 0), (0, 1, 0)]]
multiplicities = [1, 1, 2], integral points = [[(0, 0, 0)], [(0, 0,
0)], [(0, 0, 0), (1, 1, 1)]]
J_tau = (2*s + 5)/(2*s + 3)^2 , dim_tau!*Vol(tau) = 1


tau2: dim 1,  vertices = [(0, 1, 1)],  rays = [(0, 0, 1)]
generators of cone = [(1, 0, 0), (1, 2, 0)], partition into
simplicial cones = [[(1, 0, 0), (1, 2, 0)]]
multiplicities = [2], integral points = [[(0, 0, 0), (1, 1, 0)]]
J_tau = 2/(2*s + 3) , dim_tau!*Vol(tau) = 0
```

```
tau3: dim 1,  vertices = [(0, 1, 1)],  rays = [(0, 1, 0)]
generators of cone = [(1, 0, 0), (1, 0, 2)], partition into
simplicial cones = [[(1, 0, 0), (1, 0, 2)]]
multiplicities = [2], integral points = [[(0, 0, 0), (1, 0, 1)]]
J_tau = 2/(2*s + 3) , dim_tau!*Vol(tau) = 0


tau4: dim 1,  vertices = [(2, 0, 0)],  rays = [(0, 0, 1)]
generators of cone = [(0, 1, 0), (1, 2, 0)], partition into
simplicial cones = [[(0, 1, 0), (1, 2, 0)]]
multiplicities = [1], integral points = [[(0, 0, 0)]]
J_tau = 1/(2*s + 3) , dim_tau!*Vol(tau) = 0


tau5: dim 1,  vertices = [(2, 0, 0)],  rays = [(0, 1, 0)]
generators of cone = [(0, 0, 1), (1, 0, 2)], partition into
simplicial cones = [[(0, 0, 1), (1, 0, 2)]]
multiplicities = [1], integral points = [[(0, 0, 0)]]
J_tau = 1/(2*s + 3) , dim_tau!*Vol(tau) = 0


tau6: dim 1,  vertices = [(0, 1, 1), (2, 0, 0)],  rays = []
generators of cone = [(1, 0, 2), (1, 2, 0)], partition into
simplicial cones = [[(1, 0, 2), (1, 2, 0)]]
multiplicities = [2], integral points = [[(0, 0, 0), (1, 1, 1)]]
J_tau = 2/(2*s + 3)^2 , dim_tau!*Vol(tau) = 1


tau7: dim 1,  vertices = [(2, 0, 0)],  rays = [(1, 0, 0)]
generators of cone = [(0, 1, 0), (0, 0, 1)], partition into
simplicial cones = [[(0, 1, 0), (0, 0, 1)]]
multiplicities = [1], integral points = [[(0, 0, 0)]]
J_tau = 1 , dim_tau!*Vol(tau) = 0


tau8: dim 2,  vertices = [(0, 1, 1)],  rays = [(0, 0, 1), (0, 1, 0)]
generators of cone = [(1, 0, 0)], partition into simplicial cones =
[[(1, 0, 0)]]
multiplicities = [1], integral points = [[(0, 0, 0)]]
J_tau = 1 , dim_tau!*Vol(tau) = 0


tau9: dim 2,  vertices = [(2, 0, 0)],  rays = [(0, 0, 1), (1, 0, 0)]
generators of cone = [(0, 1, 0)], partition into simplicial cones =
[[(0, 1, 0)]]
multiplicities = [1], integral points = [[(0, 0, 0)]]
J_tau = 1 , dim_tau!*Vol(tau) = 0


tau10: dim 2,  vertices = [(2, 0, 0)],  rays = [(0, 1, 0), (1, 0,
0)]
generators of cone = [(0, 0, 1)], partition into simplicial cones =
[[(0, 0, 1)]]
multiplicities = [1], integral points = [[(0, 0, 0)]]
J_tau = 1 , dim_tau!*Vol(tau) = 0


tau11: dim 2,  vertices = [(0, 1, 1), (2, 0, 0)],  rays = [(0, 1,
0)]
generators of cone = [(1, 0, 2)], partition into simplicial cones =
[[(1, 0, 2)]]
multiplicities = [1], integral points = [[(0, 0, 0)]]
J_tau = 1/(2*s + 3) , dim_tau!*Vol(tau) = 0


tau12: dim 2,  vertices = [(0, 1, 1), (2, 0, 0)],  rays = [(0, 0,
```

```
1)]
generators of cone = [(1, 2, 0)], partition into simplicial cones =
[[(1, 2, 0)]]
multiplicities = [1], integral points = [[(0, 0, 0)]]
J_tau = 1/(2*s + 3) , dim_tau!*Vol(tau) = 0


(s + 3)/((s + 1)*(2*s + 3))
```

Example 11: $x^2 + yz$

$d = 2$:

```
zex11 = zex10
```

```
zex11.give_expected_pole_info(d = 2)
```

```
-3/2 with expected order: 2
The responsible face(s) of maximal dimension is/are:
        tau6: dim 1,  vertices = [(0, 1, 1), (2, 0, 0)],  rays = []
       generators of cone = [(1, 0, 2), (1, 2, 0)], partition into
simplicial cones = [[(1, 0, 2), (1, 2, 0)]]
```

```
zex11.topological_zeta(d = 2, info = True)
```

```
tau1: dim 0,  vertices = [(2, 0, 0)],  rays = []
generators of cone = [(0, 1, 0), (0, 0, 1), (1, 0, 2), (1, 2, 0)],
partition into simplicial cones = [[(1, 0, 2), (0, 1, 0)], [(1, 0,
2), (0, 0, 1), (0, 1, 0)], [(1, 0, 2), (1, 2, 0), (0, 1, 0)]]
multiplicities = [1, 1, 2], integral points = [[(0, 0, 0)], [(0, 0,
0)], [(0, 0, 0), (1, 1, 1)]]
J_tau = (2*s + 5)/(2*s + 3)^2 , dim_tau!*Vol(tau) = 1


tau4: dim 1,  vertices = [(2, 0, 0)],  rays = [(0, 0, 1)]
generators of cone = [(0, 1, 0), (1, 2, 0)], partition into
simplicial cones = [[(0, 1, 0), (1, 2, 0)]]
multiplicities = [1], integral points = [[(0, 0, 0)]]
J_tau = 1/(2*s + 3) , dim_tau!*Vol(tau) = 0


tau5: dim 1,  vertices = [(2, 0, 0)],  rays = [(0, 1, 0)]
generators of cone = [(0, 0, 1), (1, 0, 2)], partition into
simplicial cones = [[(0, 0, 1), (1, 0, 2)]]
multiplicities = [1], integral points = [[(0, 0, 0)]]
J_tau = 1/(2*s + 3) , dim_tau!*Vol(tau) = 0


tau6: dim 1,  vertices = [(0, 1, 1), (2, 0, 0)],  rays = []
generators of cone = [(1, 0, 2), (1, 2, 0)], partition into
simplicial cones = [[(1, 0, 2), (1, 2, 0)]]
multiplicities = [2], integral points = [[(0, 0, 0), (1, 1, 1)]]
J_tau = 2/(2*s + 3)^2 , dim_tau!*Vol(tau) = 1


tau7: dim 1,  vertices = [(2, 0, 0)],  rays = [(1, 0, 0)]
generators of cone = [(0, 1, 0), (0, 0, 1)], partition into
simplicial cones = [[(0, 1, 0), (0, 0, 1)]]
multiplicities = [1], integral points = [[(0, 0, 0)]]
J_tau = 1 , dim_tau!*Vol(tau) = 0


tau8: dim 2,  vertices = [(0, 1, 1)],  rays = [(0, 0, 1), (0, 1, 0)]
generators of cone = [(1, 0, 0)], partition into simplicial cones =
[[(1, 0, 0)]]
multiplicities = [1], integral points = [[(0, 0, 0)]]
J_tau = 1 , dim_tau!*Vol(tau) = 0
```

```
tau9: dim 2,  vertices = [(2, 0, 0)],  rays = [(0, 0, 1), (1, 0, 0)]
generators of cone = [(0, 1, 0)], partition into simplicial cones =
[[(0, 1, 0)]]
multiplicities = [1], integral points = [[(0, 0, 0)]]
J_tau = 1 , dim_tau!*Vol(tau) = 0


tau10: dim 2,  vertices = [(2, 0, 0)],  rays = [(0, 1, 0), (1, 0,
0)]
generators of cone = [(0, 0, 1)], partition into simplicial cones =
[[(0, 0, 1)]]
multiplicities = [1], integral points = [[(0, 0, 0)]]
J_tau = 1 , dim_tau!*Vol(tau) = 0


tau11: dim 2,  vertices = [(0, 1, 1), (2, 0, 0)],  rays = [(0, 1,
0)]
generators of cone = [(1, 0, 2)], partition into simplicial cones =
[[(1, 0, 2)]]
multiplicities = [1], integral points = [[(0, 0, 0)]]
J_tau = 1/(2*s + 3) , dim_tau!*Vol(tau) = 0


tau12: dim 2,  vertices = [(0, 1, 1), (2, 0, 0)],  rays = [(0, 0,
1)]
generators of cone = [(1, 2, 0)], partition into simplicial cones =
[[(1, 2, 0)]]
multiplicities = [1], integral points = [[(0, 0, 0)]]
J_tau = 1/(2*s + 3) , dim_tau!*Vol(tau) = 0


1/(2*s + 3)
```

Example 12: $x^2y^2z + xyz^2$

$d = 2$:

```
zex12 = ZetaFunctions(R(x^2*y^2*z + x*y*z^2))
```

```
zex12.give_expected_pole_info(d = 2)
```

```
There will be no poles for the (local) topological zeta function
(with d = 2) of x^2*y^2*z + x*y*z^2.
```

```
zex12.topological_zeta(d = 2)
```

```
0
```

Example 14: $xyz + uvw + xyw + zuv$

```
S2.<x,y,z,u,v,w> = QQ[]
zex14 = ZetaFunctions(x*y*z + u*v*w + x*y*w + z*u*v)
```

```
zex14.give_info_newton()
```

```
Newton's polyhedron of x*y*z + z*u*v + x*y*w + u*v*w:
        support points = [(1, 1, 1, 0, 0, 0), (0, 0, 1, 1, 1, 0), (1, 1, 0,
0, 0, 1), (0, 0, 0, 1, 1, 1)]
        vertices = [(0, 0, 0, 1, 1, 1), (0, 0, 1, 1, 1, 0), (1, 1, 0, 0, 0,
1), (1, 1, 1, 0, 0, 0)]
        number of proper faces = 203
```

```
        Facet 1: z + w - 1 >= 0
        Facet 2: w >= 0
        Facet 3: u >= 0
        Facet 4: v >= 0
        Facet 5: x + v - 1 >= 0
        Facet 6: x + u - 1 >= 0
        Facet 7: y + u - 1 >= 0
        Facet 8: y + v - 1 >= 0
        Facet 9: y >= 0
        Facet 10: x >= 0
        Facet 11: z >= 0
```

```
zex14.give_expected_pole_info()
```

```
The candidate poles of the (local) topological zeta function (with d
= 1) of x*y*z + z*u*v + x*y*w + u*v*w in function of s are:

-2 with expected order: 4
The responsible face of maximal dimension is ``tau_0`` = minimal
face who intersects with the diagonal of ambient space:
         tau178: dim 2,  vertices = [(0, 0, 0, 1, 1, 1), (0, 0, 1, 1, 1,
0), (1, 1, 0, 0, 0, 1), (1, 1, 1, 0, 0, 0)],  rays = []
        generators of cone = [(0, 0, 1, 0, 0, 1), (1, 0, 0, 0, 1, 0), (1,
0, 0, 1, 0, 0), (0, 1, 0, 1, 0, 0), (0, 1, 0, 0, 1, 0)], partition
into simplicial cones = [[(0, 0, 1, 0, 0, 1), (1, 0, 0, 0, 1, 0),
(0, 1, 0, 1, 0, 0)], [(0, 0, 1, 0, 0, 1), (1, 0, 0, 0, 1, 0), (1, 0,
0, 1, 0, 0), (0, 1, 0, 1, 0, 0)], [(0, 0, 1, 0, 0, 1), (0, 1, 0, 0,
1, 0), (0, 1, 0, 1, 0, 0), (1, 0, 0, 0, 1, 0)]]

-1 with expected order: 1
(If all Vol(tau) are 0, where tau runs through the selected faces
that are no vertices, then the expected order of -1 is 0).
```

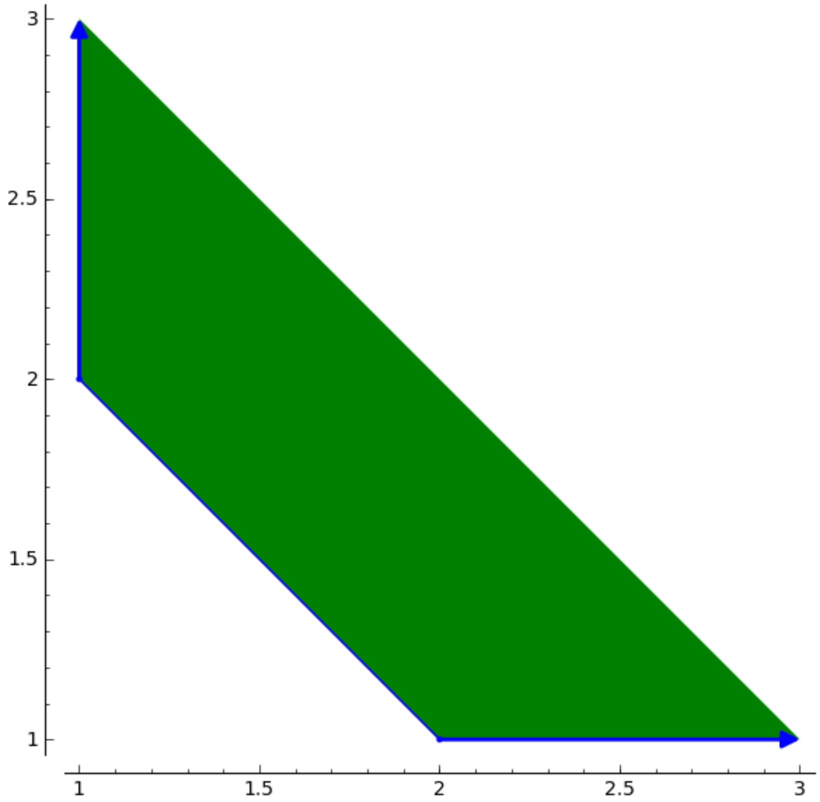
```
zex14.topological_zeta()
```

```
The formula for Topological Zeta function is not valid:
The polynomial is degenerated at least with respect to the face tau
= {dim 2,  vertices = [(0, 0, 0, 1, 1, 1), (0, 0, 1, 1, 1, 0), (1,
1, 0, 0, 0, 1), (1, 1, 1, 0, 0, 0)],  rays = []} over the complex
numbers!
NaN
```

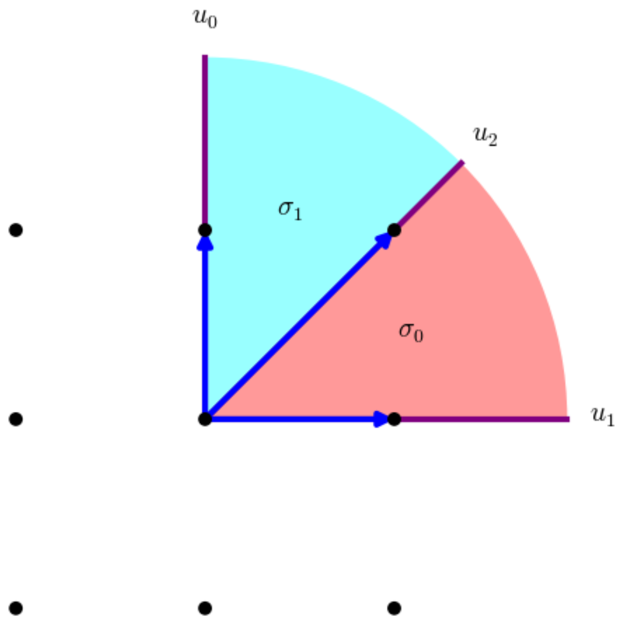
Example 15: $xy^3 + xy^2 + x^2y$

```
R2.<x,y> = QQ[]
zex15 = ZetaFunctions(x*y^3 + x*y^2 + x^2*y)
```

```
zex15.newton_plot()
```




```
zex15.cones_plot()
```



Local:

```
zex15.give_expected_pole_info(local = True)
```

```
The candidate poles of the (local) topological zeta function (with d
= 1) of x*y^3 + x^2*y + x*y^2 in function of s are:
```

```
-2/3 with expected order: 1
The responsible face of maximal dimension is ``tau_0`` = minimal
face who intersects with the diagonal of ambient space:
         tau4: dim 1,  vertices = [(1, 2), (2, 1)],  rays = []
        generators of cone = [(1, 1)], partition into simplicial cones =
[[(1, 1)]]

-1 with expected order: 1
The responsible face(s) of maximal dimension is/are:
         tau1: dim 0,  vertices = [(2, 1)],  rays = []
        generators of cone = [(0, 1), (1, 1)], partition into simplicial
cones = [[(0, 1), (1, 1)]]

         tau0: dim 0,  vertices = [(1, 2)],  rays = []
        generators of cone = [(1, 0), (1, 1)], partition into simplicial
cones = [[(1, 0), (1, 1)]]
```

```
zex15.topological_zeta(local = True, info = True)
```

```
tau0: dim 0,  vertices = [(1, 2)],  rays = []
generators of cone = [(1, 0), (1, 1)], partition into simplicial
cones = [[(1, 0), (1, 1)]]
multiplicities = [1], integral points = [[(0, 0)]]
J_tau = 1/((s + 1)*(3*s + 2)) , dim_tau!*Vol(tau) = 1


tau1: dim 0,  vertices = [(2, 1)],  rays = []
generators of cone = [(0, 1), (1, 1)], partition into simplicial
cones = [[(0, 1), (1, 1)]]
multiplicities = [1], integral points = [[(0, 0)]]
J_tau = 1/((s + 1)*(3*s + 2)) , dim_tau!*Vol(tau) = 1


tau4: dim 1,  vertices = [(1, 2), (2, 1)],  rays = []
generators of cone = [(1, 1)], partition into simplicial cones =
[[(1, 1)]]
multiplicities = [1], integral points = [[(0, 0)]]
J_tau = 1/(3*s + 2) , dim_tau!*Vol(tau) = 1


-(s - 2)/((s + 1)*(3*s + 2))
```

Example 19: $x_1x_2x_3^2x_4 + x_1x_2^2x_3x_4 + x_1^2x_2x_3x_4^2$

```
T.<x_1,x_2,x_3,x_4> = QQ[]
zex19 = ZetaFunctions(x_1*x_2*x_3^2*x_4 + x_1*x_2^2*x_3*x_4 + x_1^2*x_2*x_3*x_4^2)
```

```
zex19.give_info_newton()
```

```
Newton's polyhedron of x_1^2*x_2*x_3*x_4^2 + x_1*x_2^2*x_3*x_4 +
x_1*x_2*x_3^2*x_4:
        support points = [(2, 1, 1, 2), (1, 2, 1, 1), (1, 1, 2, 1)]
        vertices = [(1, 1, 2, 1), (1, 2, 1, 1), (2, 1, 1, 2)]
        number of proper faces = 33
        Facet 1: x_2 - 1 >= 0
        Facet 2: x_3 - 1 >= 0
        Facet 3: x_1 - 1 >= 0
        Facet 4: x_4 - 1 >= 0
        Facet 5: x_2 + x_3 + x_4 - 4 >= 0
        Facet 6: x_1 + x_2 + x_3 - 4 >= 0
```

```
zex19.give_expected_pole_info()
```

```
The candidate poles of the (local) topological zeta function (with d
= 1) of x_1^2*x_2*x_3*x_4^2 + x_1*x_2^2*x_3*x_4 + x_1*x_2*x_3^2*x_4
in function of s are:

-3/4 with expected order: 2
The responsible face of maximal dimension is ``tau_0`` = minimal
```

```
face who intersects with the diagonal of ambient space:
        tau26: dim 2,  vertices = [(1, 1, 2, 1), (1, 2, 1, 1), (2, 1, 1,
2)],  rays = []
       generators of cone = [(0, 1, 1, 1), (1, 1, 1, 0)], partition into
simplicial cones = [[(0, 1, 1, 1), (1, 1, 1, 0)]]

-1 with expected order: 3
The responsible face(s) of maximal dimension is/are:
        tau5: dim 1,  vertices = [(1, 1, 2, 1)],  rays = [(0, 0, 1, 0)]
       generators of cone = [(0, 1, 0, 0), (1, 0, 0, 0), (0, 0, 0, 1)],
partition into simplicial cones = [[(0, 1, 0, 0), (1, 0, 0, 0), (0,
0, 0, 1)]]

        tau9: dim 1,  vertices = [(1, 2, 1, 1)],  rays = [(0, 1, 0, 0)]
       generators of cone = [(0, 0, 1, 0), (1, 0, 0, 0), (0, 0, 0, 1)],
partition into simplicial cones = [[(0, 0, 1, 0), (1, 0, 0, 0), (0,
0, 0, 1)]]
```

```
zex19.topological_zeta()
```

```
(s^3 - 5*s^2 + 6*s + 9)/((s + 1)^3*(4*s + 3)^2)
```

Example 21: $x_1^2 + x_2^3x_4^3 + x_3^3x_5^3$

```
T2.<x_1,x_2,x_3,x_4,x_5> = QQ[]
zex21 = ZetaFunctions(x_1^2 + x_2^3*x_4^3 + x_3^3*x_5^3)
```

```
zex21.give_info_newton()
```

```
Newton's polyhedron of x_2^3*x_4^3 + x_3^3*x_5^3 + x_1^2:
       support points = [(0, 3, 0, 3, 0), (0, 0, 3, 0, 3), (2, 0, 0, 0,
0)]
       vertices = [(0, 0, 3, 0, 3), (0, 3, 0, 3, 0), (2, 0, 0, 0, 0)]
       number of proper faces = 85
       Facet 1: x_2 >= 0
       Facet 2: x_4 >= 0
       Facet 3: x_3 >= 0
       Facet 4: x_5 >= 0
       Facet 5: x_1 >= 0
       Facet 6: 3*x_1 + 2*x_3 + 2*x_4 - 6 >= 0
       Facet 7: 3*x_1 + 2*x_4 + 2*x_5 - 6 >= 0
       Facet 8: 3*x_1 + 2*x_2 + 2*x_5 - 6 >= 0
       Facet 9: 3*x_1 + 2*x_2 + 2*x_3 - 6 >= 0
```

```
zex21.give_expected_pole_info()
```

```
The candidate poles of the (local) topological zeta function (with d
= 1) of x_2^3*x_4^3 + x_3^3*x_5^3 + x_1^2 in function of s are:

-7/6 with expected order: 3
The responsible face of maximal dimension is ``tau_0`` = minimal
face who intersects with the diagonal of ambient space:
        tau57: dim 2,  vertices = [(0, 0, 3, 0, 3), (0, 3, 0, 3, 0), (2,
0, 0, 0, 0)],  rays = []
       generators of cone = [(3, 0, 2, 2, 0), (3, 0, 0, 2, 2), (3, 2, 0,
0, 2), (3, 2, 2, 0, 0)], partition into simplicial cones = [[(3, 0,
2, 2, 0), (3, 2, 0, 0, 2)], [(3, 0, 0, 2, 2), (3, 2, 0, 0, 2), (3,
0, 2, 2, 0)], [(3, 2, 0, 0, 2), (3, 2, 2, 0, 0), (3, 0, 2, 2, 0)]]

-1 with expected order: 1
(If all Vol(tau) are 0, where tau runs through the selected faces
that are no vertices, then the expected order of -1 is 0).
```

```
zex21.topological_zeta()
```

```
(108*s^3 + 456*s^2 + 647*s + 343)/((s + 1)*(6*s + 7)^3)
```

Examples for the **Monodromy Zeta Function at the origin**

```
zexmon1 = ZetaFunctions(R2(y^7+x^2*y^5+x^5*y^3))
zexmon1.monodromy_zeta(char = True)
```

```
The characteristic polynomial of the monodromy is (T - 1)^3*(T^6 +
T^5 + T^4 + T^3 + T^2 + T + 1)*(T^18 + T^17 + T^16 + T^15 + T^14 +
T^13 + T^12 + T^11 + T^10 + T^9 + T^8 + T^7 + T^6 + T^5 + T^4 + T^3
+ T^2 + T + 1)

1/((t^7 - 1)*(t^19 - 1))
```

```
zexmon2 = ZetaFunctions(R(x*y + z^3))
zexmon2.monodromy_zeta(char = True)
```

```
The characteristic polynomial of the monodromy is T^2 + T + 1

-t^3 + 1
```

```
zexmon3 = ZetaFunctions(R((3*x+5*z)*(x+2*z)+y^3))
zexmon3.monodromy_zeta(char = True)
```

```
The characteristic polynomial of the monodromy is T^2 + T + 1

-t^3 + 1
```

```
zexmon4 = ZetaFunctions(R(x*(y+x)+x^2*z+z^3))
zexmon4.monodromy_zeta(char = True)
```

```
The characteristic polynomial of the monodromy is T^2 + T + 1

-t^3 + 1
```

```
zexmon4 = ZetaFunctions(R(x*y*(x+y)+z^4))
zexmon4.monodromy_zeta(char = True)
```

```
The characteristic polynomial of the monodromy is (T + 1)^2*(T^2 +
1)^2*(T^2 - T + 1)*(T^4 - T^2 + 1)

-(t^4 - 1)*(t^12 - 1)/(t^3 - 1)
```