Combinatorics and topology of line arrangements via configurations of points

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International School on Singularities and Lipschitz Geometry, June 11th to 22nd, 2018

Some historical facts in topology of Line arrangements and Zariski pairs

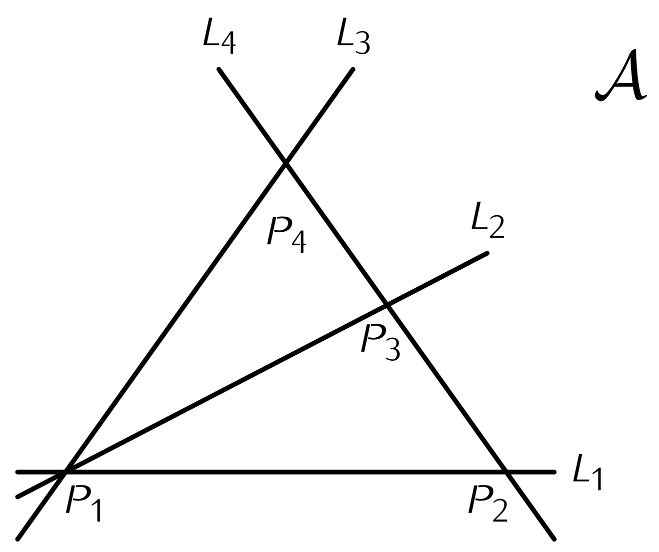
Line arrangements in $\mathbb{C}P^2$ are classically studied as **simpler case of singular plane algebraic curves**

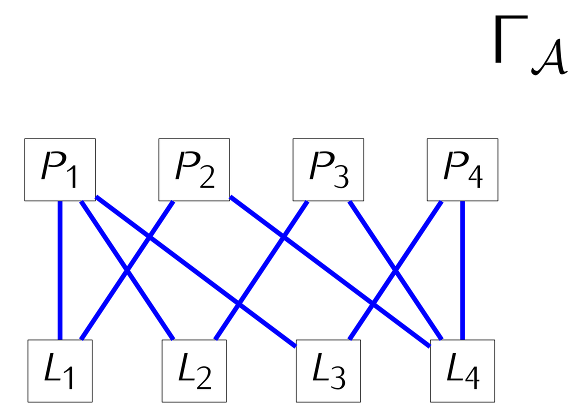
Definition

A (complex) *line arrangement* $\mathcal{A}$ is a finite collection of distinct lines in $\mathbb{C}P^2$,

$$\mathcal{A} = \{L_0, L_1, \ldots, L_n\}.$$

$\mathcal{A}$ is *real complexified* if there exists a system of coordinates of $\mathbb{C}P^2$ such that any $L \in \mathcal{A}$ is defined by a $\mathbb{R}$-linear form.





The combinatorics is expressed by the associated *incidence graph* $\Gamma_{\mathcal{A}}$, with **vertices composed by the lines and singular points, joined by an edge if $P \in L$.**

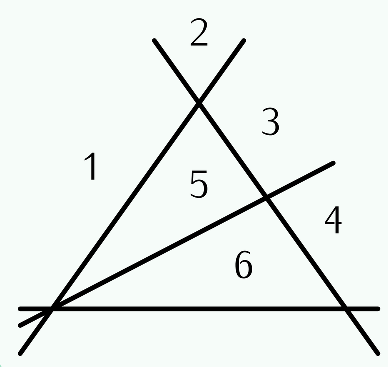
Main Question: What is the influence of the combinatorics $\Gamma_{\mathcal{A}}$ over the embedded topology of $\mathcal{A}$?

Topology of $\mathcal{A}$: homeomorphism type of the pair $(\mathbb{C}P^2, \mathcal{A})$.
Up to 9 lines, the topoloy of $\mathcal{A}$ is determined by $\Gamma_{\mathcal{A}}$. We are interested in produce counterexamples:

Definition

A *Zariski pair* is a couple of line arrangements $(\mathcal{A}_1, \mathcal{A}_2)$ with:

- same combinatorics: $\Gamma_{\mathcal{A}_1} \sim \Gamma_{\mathcal{A}_2}$.
- different topologies: $(\mathbb{C}P^2, \mathcal{A}_1) \not\approx (\mathbb{C}P^2, \mathcal{A}_2)$.



Example over $\mathbb{R}$. Consider that $\mathcal{A} \subset \mathbb{R}P^2$ and let $\mathcal{R}_{\mathcal{A}} = \#$regions in $\mathbb{R}P^2 \setminus \mathcal{A}$. **Is this number determined by the combinatorics? Yes**, Zaslavsky in 1975 gives the following formula:

$$\mathcal{R}_{\mathcal{A}} = 1 + \sum_{k \geq 2} n_k \cdot (k-1)$$

where n_k is the number of singularities of multiplicity k.

At this moment, **there exist only three known examples of Zariski pairs**: two pure complex arrangements [5, 4], distinguished by the fundamental group of the complement $\pi_1(\mathbb{C}P^2 \setminus \mathcal{A})$, and a real complexified one [1], detected by using other topological tools. **All of them are defined over a non-trivial number field $\mathbb{Q}(\alpha)$, with $\alpha \in \overline{\mathbb{Q}} \setminus \mathbb{Q}$.**

In general: topological invariants are difficult to compute and differentiate when n increases (a computer assistant is needed). In the case of $\pi_1(\mathbb{C}P^2 \setminus \mathcal{A})$, in general to establish if two such groups are different is a very hard computational problem. **All of the previous examples of Zariski pairs needed computer calculations at some point.**

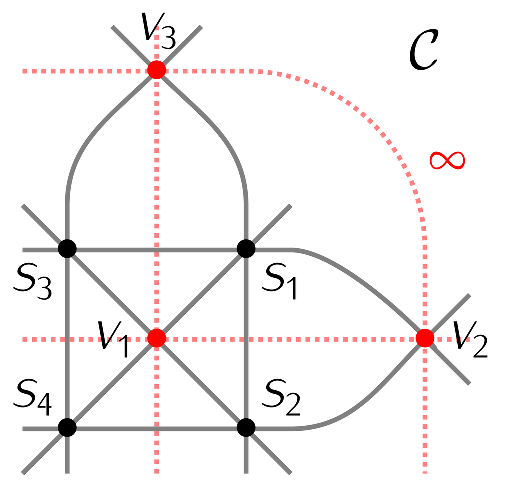
Open Questions:

- Is there a way to detect Zariski pairs without using computer calculations?
- A more geometrical way to construct Zariski pairs?
- Is $\pi_1(\mathbb{C}P^2 \setminus \mathcal{A})$ combinatorially determined for $\mathcal{A}$ real complexified?
- Could Zarsiki pairs be realized over $\mathbb{Q}$?

Our Work: Configurations of points and counting parities

We take in the *dual real plane* $\check{\mathbb{R}P}^2 = \{L \mid L \subset \mathbb{R}P^2 \text{ a line}\}$:

- $\mathcal{V} = \{V_1, V_2, V_3\}$ points in general position called *vertices*,
- $\mathcal{S} = \{S_1, \ldots, S_n\}$ points called *surrounding-points*,
- $\mathcal{L} = \{\overline{SV} \mid S \in \mathcal{S},\ V \in \mathcal{V}\}$ collection of lines.



Definition

The tuple $\mathcal{C} = (\mathcal{V}, \mathcal{S}, \mathcal{L})$ is a *planar* $(3,2)$-*configuration* if:

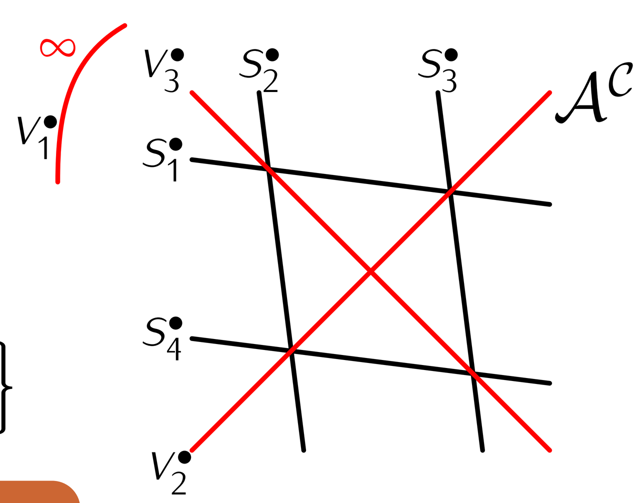
1. $\forall V_i, V_j \in \mathcal{V} : \mathcal{S} \cap \overline{V_i V_j} = \emptyset$,
2. $\forall L \in \mathcal{L} : \#\mathcal{S} \cap L \equiv 0 \bmod 2$.

Combinatorics: (nontrivial) collinearity relations between points $\mathcal{V} \sqcup \mathcal{S}$ in $\mathbb{R}P^2$.
Using configurations of points in the dual space, we can establish a **natural dictionary** between configurations of points in $\check{\mathbb{R}P}^2$ and real complexified arrangements in $\mathbb{C}P^2 = \mathbb{R}P^2 \otimes \mathbb{C}$:

$$\check{\mathbb{R}P}^2 \xrightarrow{P \mapsto P^\bullet = P^\vee \otimes \mathbb{C}} \mathbb{C}P^2 = \mathbb{R}P^2 \otimes \mathbb{C}$$
$$\mathcal{V} \longrightarrow \mathcal{A}^{\mathcal{V}} = \{V_1^\bullet, V_2^\bullet, V_3^\bullet\}$$
$$\mathcal{S} \longrightarrow \mathcal{A}^{\mathcal{S}} = \{S_1^\bullet, \ldots, S_n^\bullet\}$$

Dual Arrangement $\boxed{\mathcal{A}^{\mathcal{C}} = \mathcal{A}^{\mathcal{V}} \sqcup \mathcal{A}^{\mathcal{S}}}$

$$\left\{ \begin{array}{c} \text{Combinatorics of } \mathcal{C} \\ \text{(colinearities)} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Combinatorics of } \mathcal{A}^{\mathcal{C}} \\ \text{(incidences)} \end{array} \right\}$$

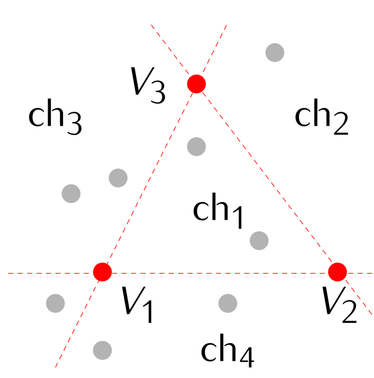


Definition

$\mathcal{C}$ is *stable* if for any $\phi \in \mathrm{Aut}(\mathcal{V} \sqcup \mathcal{S})$ resp. collinearity, we have $\phi(\mathcal{V}) = \mathcal{V}$.

Take $\mathcal{C} = (\mathcal{V}, \mathcal{S}, \mathcal{L})$ a planar $(3,2)$-configuration: the vertices V_1, V_2, V_3 define a partition of $\mathbb{R}P^2$ in 4 chambers.

We are interested on **counting the parity of points contained at any chamber**. In fact, we will prove that this number is a **topological invariant** of the associated line arrangement.



Our Work: New Zariski pairs obtained "by hand"

Definition

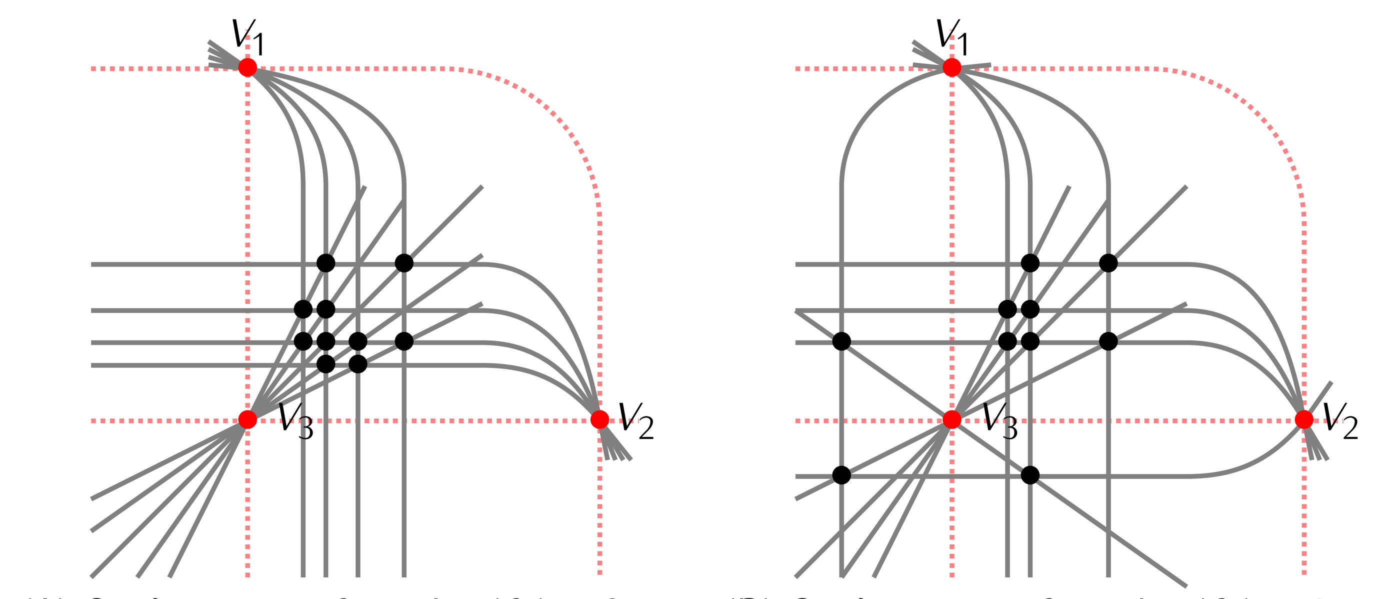
The *chamber weight* of $\mathcal{C}$ is the value

$$\tau(\mathcal{C}) = \#\mathcal{S} \cap \mathrm{ch}_i \mod 2$$

and does not depend on the choice of ch$_i$.

Theorem ([4], Guerville-Ballé, ___)

If $\mathcal{C}$ is stable, then $\tau(\mathcal{C})$ is a topological invariant of $(\mathbb{C}P^2, \mathcal{A}^{\mathcal{C}})$.



(A) Configuration $\mathcal{C}_1$ with $\tau(\mathcal{C}_1) = 0$. (B) Configuration $\mathcal{C}_2$ with $\tau(\mathcal{C}_2) = 1$.

Theorem ([4], Guerville-Ballé, ___)

The configurations $\mathcal{C}_1$ and $\mathcal{C}_2$ are stables, defined over $\mathbb{Q}$, and have the same combinatorics. Moreover, the dual arrangements $(\mathcal{A}^{\mathcal{C}_1}, \mathcal{A}^{\mathcal{C}_2})$ form a Zariski pair.

Proof. Just counting points in any of the chambers of $\mathcal{C}_1$ and $\mathcal{C}_2$!

Using the previous method, we present a total of **10 new examples of Zariski pairs** in [4]. Moreover,

Theorem ([2], Artal, Guerville-Ballé, ___)

The fundamental groups $\pi_1\left(\mathbb{C}P^2 \setminus \mathcal{A}^{\mathcal{C}_1}\right)$ and $\pi_1\left(\mathbb{C}P^2 \setminus \mathcal{A}^{\mathcal{C}_2}\right)$ are not isomorphic.

References

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