A new formula for the motivic and topological zeta functions from $\mathbb{Q}$-resolution of singularities

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Joint work with
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Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a holomorphic germ. Take $(D, 0)$ defined by f.

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$$\mathrm{div}(h^* f) = \sum_{i \in S} N_i E_i \quad \text{and} \quad \mathrm{div}(h^*(\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n)) = \sum_{i \in S} (\nu_i - 1) E_i.$$

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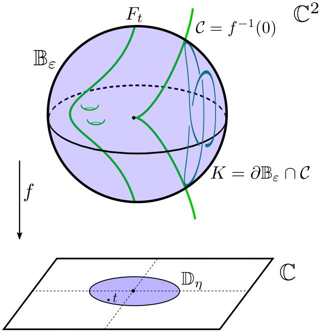
The Monodromy conjecture

- Consider $F_t = \{f = t\} \cap \mathbb{B}_\varepsilon$ being the Milnor fiber of f at the origin, for some $0 < \eta \ll \varepsilon \ll 1$.

- The monodromy action $H^\bullet(F_t;\mathbb{C}) \xrightarrow{\sigma^*} H^\bullet(F_t;\mathbb{C})$.



Conjecture (Igusa, Denef-Loeser)

If $s_0 \in \mathbb{C}$ is a pole of $Z_{\mathrm{top},0}(f;s)$, then $e^{2\pi s_0}$ is an eigenvalue of some $H^i(F_t;\mathbb{C}) \xrightarrow{\sigma^*} H^i(F_t;\mathbb{C})$, at some closed point $x_0 \in f^{-1}(0)$ of the origin.

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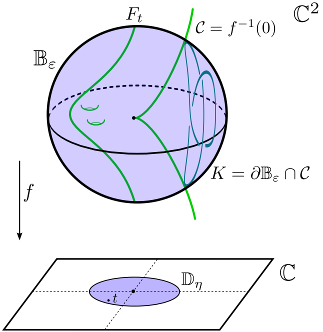

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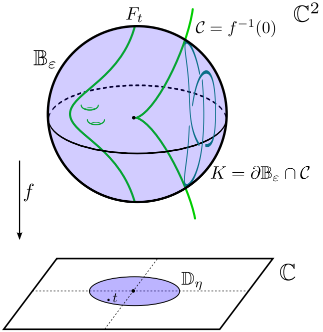

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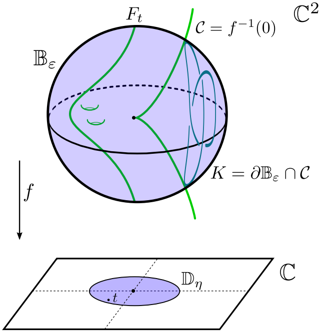

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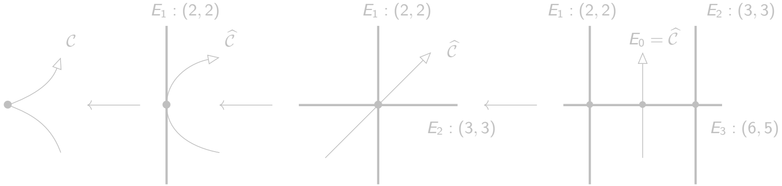

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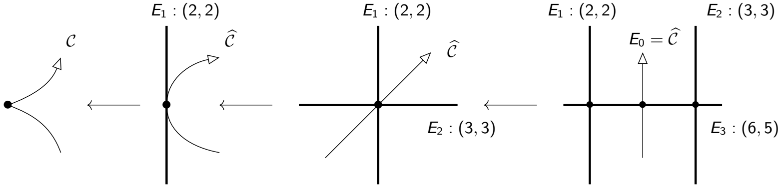

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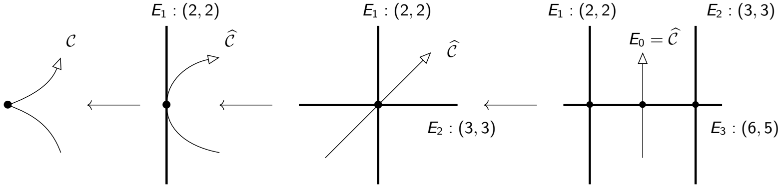

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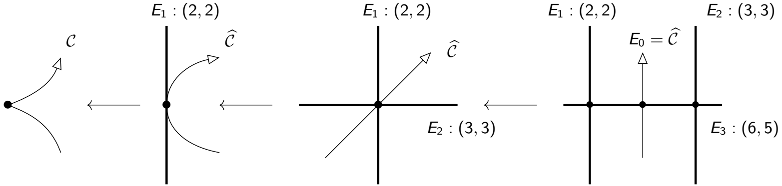

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EXAMPLE 1: take $\omega = (q_0, \ldots, q_n) \in \mathbb{Z}^n$ coprimes. The *ω-weighted projective space*:

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- $E \simeq \mathbb{P}^1_{(p,q)}$
- E contains singularities!

V-manifolds and $\mathbb{Q}$-normal crossings

Definition

A complex analytic manifold Y is called a *V-manifold* if $Y = \bigcup_k U_k$ such that each open $U_k \simeq \mathbb{C}^n / G_k$, for some finite $G_k \subset \mathrm{GL}_n(\mathbb{C})$.

EXAMPLE 1: take $\omega = (q_0, \ldots, q_n) \in \mathbb{Z}^n$ coprimes. The *ω-weighted projective space*:

$$\mathbb{P}^2_\omega = \frac{\mathbb{C}^{n+1} \setminus 0}{(x_0, \ldots, x_n) \underset{\lambda \in \mathbb{C}^*}{\sim} (\lambda^{q_0} x_0, \ldots, \lambda^{q_n} x_n)}$$

$\mathbb{P}^2_\omega = \bigcup_{j=0}^n U_j$, where $U_j \simeq \frac{1}{q_j}(q_0, \ldots, \widehat{q}_j, \ldots, q_n) = \mathbb{C}^n / C_{q_j}$.

EXAMPLE 2: The (p,q)-blowing up of the plane, $\gcd(p,q) = 1$:

$$\left\{ ((x,y), [\lambda : \mu]) \in \mathbb{C}^2 \times \mathbb{P}^1_{(p,q)} \mid (x,y) \in \overline{[\lambda : \mu]} \right\}$$

$$\begin{array}{ccccc} E \subset & \widehat{\mathbb{C}}^2_{(p,q)} \simeq & \frac{1}{p}(-1,q) & \cup & \frac{1}{q}(p,-1) \\ & \Big\downarrow \pi_{(p,q)} & [(x,y)] & & [(x,y)] \\ & & \big\downarrow & & \big\downarrow \\ \{0\} \subset & \mathbb{C}^2 & (x^p, x^q y) & & (xy^p, y^q) \end{array}$$

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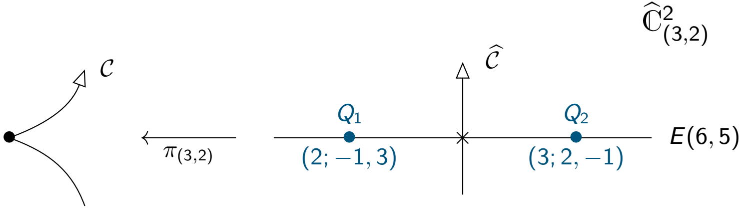
Weighted (p, q)-blowing up of the plane

Cusp: $f(x, y) = y^2 - x^3$



$\mathrm{Sing}\left(\widehat{\mathbb{C}}^2_{(3,2)}\right) = \{Q_1, Q_2\}$

Definition

A hypersurface $D \subset Y$ has $\mathbb{Q}$-*normal crossings* if it is locally isomorphic to

$$(H_1 \cup \cdots \cup H_m)/G$$

where $m \leq \dim Y$, H_i are hyperplanes and G is finite abelian.

Definition (Steenbrink'76)

An *embedded* $\mathbb{Q}$-*resolution* of $(D, 0) \subset (\mathbb{C}^n, 0)$ is a proper analytic map $h : Y \to (\mathbb{C}^n, 0)$:

1. Y is a V-manifold with only abelian quotient singularities.
2. h is an isomorphism over $Y \setminus h^{-1}(D_{\text{sing}})$.
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REMARK: Considering the different isotropy groups $\{G_k\}_{k=0}^{r}$ acting in Y, we can define a refined stratification $Y = \bigsqcup_{I \subset S} \bigsqcup_{k=0}^{r} Y_{I,k}$:

$$Y_{I,k} = \{q \in Y_I \mid G_k \text{ is the isotropy group of } q\}.$$

$\mathbb{Q}$-resolutions of singularities

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contact order
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geometric
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Denef-Loeser Motivic zeta function :

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$(K_0(\mathrm{Var}_{\mathbb{C}}), +, \cdot)$ is the ring:

- generated by classes $[X]$ of isomorphism of complex varieties.

Motivic ideas: Grothendieck ring of varieties

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EXAMPLE : $[\mathbb{P}^1] = [\mathbb{C} \sqcup \{\infty\}] = \mathbb{L} + 1$. In fact, as $\mathbb{P}^n = \mathbb{C}^n \sqcup \mathbb{P}^{n-1}$, for $n \geq 1$,

$$[\mathbb{P}^n] = \mathbb{L}^n + \mathbb{L}^{n-1} + \cdots + \mathbb{L} + 1.$$

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$$\begin{tikzcd} & \mathbb{Z} \\ & \mathbb{Z}[t] \\ K_0(\mathrm{Var}_{\mathbb{C}}) \arrow[uur, "\chi"] \arrow[ur, "P"'] \arrow[dr, "H"] \arrow[ddr, "N_p"'] & \\ & \mathbb{Z}[u,v] \\ & \mathbb{Z} \end{tikzcd}$$

$\chi(X)$ = Euler characteristic

$P_X(t)$ = Poincaré polynomial

$H_X(u,v)$ = Hodge-Deligne polynomial

$N_p(X)$ = Number of $\mathbb{F}_p$-points

Let X be an algebraic variety.

- The space of arcs: $\mathcal{L}(X) = \{\text{pts of } X \text{ with coord. in } \mathbb{C}[[t]]\}$.
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in a completion $\mathcal{M}_{\mathbb{C}} \to \widehat{\mathcal{M}}_{\mathbb{C}}$, where

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The motivic zeta function

Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be an analytic germ. Consider $s \in \mathbb{C}$ and define:

$$\mathcal{L}(\mathbb{C}^n)_0 = \left\{\gamma \in \mathcal{L}(\mathbb{C}^n) \mid \gamma(0) = 0\right\}$$

Definition

The *(local) motivic zeta function* of f is given by

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MAIN TOOL IN MOTIVIC INTEGRATION: Consider

- $h : Y \to X$ proper map between general varieties X and Y.
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Intrinsic definition as motivic integral

If $h : Y \to \mathbb{C}^n$ is an embedded resolution of D defined by f,

$$h^{-1}(D) = \bigcup_{i \in S} E_i \quad \rightsquigarrow \quad \{(N_i, \nu_i)\}_{i \in S} \quad \text{and} \quad \{Y_I\}_{I \subset S}$$

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Studies on quotient singularities

- (Veys'91): $Z_{\text{top}}(f;s)$ for log-canonical models in $n = 2$ & using determinants of p-deformations in Igusa's p-adic zeta functions.

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Theorem (León-Cardenal, Martín-Morales, Veys, _____)

$$Z_{\mathrm{mot},0}(f; s) = \mathbb{L}^{-n} \sum_{\substack{I \subset S \\ k=0,\ldots,r}} \left[Y_{I,k} \cap h^{-1}(0)\right] \cdot S_{I,k}(\mathbb{L}) \cdot \prod_{i \in I} \frac{(\mathbb{L}-1)\mathbb{L}^{-(N_i s + \nu_i)}}{1 - \mathbb{L}^{-(N_i s+\nu_i)}}.$$

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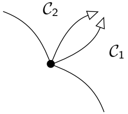
Corollary

Specializing by the Euler Characteristic:

$$Z_{\mathrm{top},0}(f; s) = \sum_{\substack{I \subset S \\ k=0,\ldots,r}} \chi\left(Y_{I,k} \cap h^{-1}(0)\right) \cdot |G_k| \cdot \prod_{i \in I} \frac{1}{N_i s + \nu_i}$$

Example: 2-branches cusp singularity

Let $f(x, y) = (x^p + y^q)(x^r + y^s)$ with $(p, q) = (u, v) = 1$ and $\frac{p}{q} < \frac{u}{v}$:



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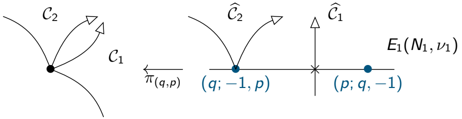

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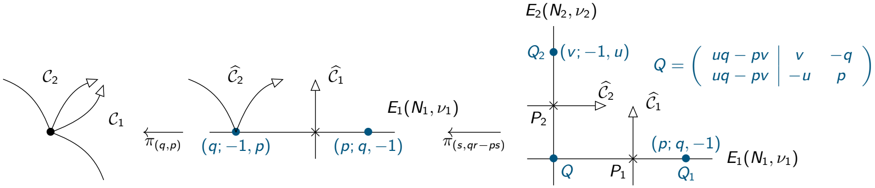

Numerical data of $h = \pi_{(v,qu-pv)} \circ \pi_{(q,p)}$:

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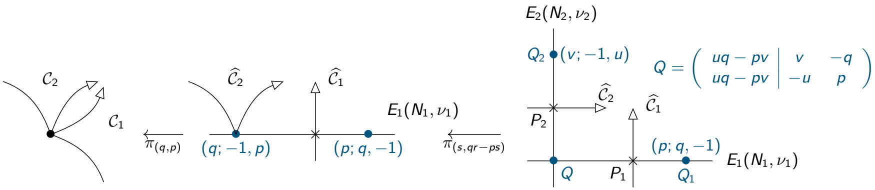

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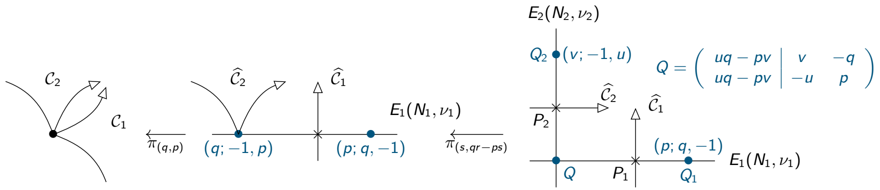

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$$E = \underbrace{E_1^* \sqcup E_2^*}_{\text{punctured lines}} \sqcup P_1 \sqcup P_2 \sqcup \underbrace{Q_1 \sqcup Q_2 \sqcup Q}_{\text{quot. sings.}}$$

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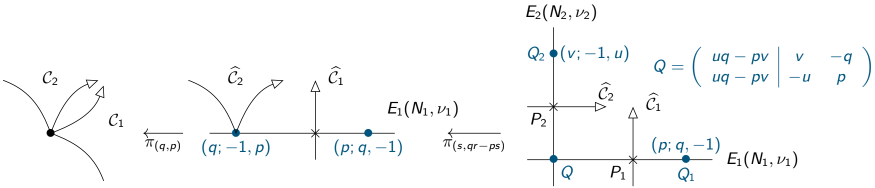

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Euler Characteristics:

$$\chi(E_1^*) = \chi(E_2^*) = \chi(\mathbb{P}^1_\omega \setminus 3\,\mathrm{pt}) = \chi(\mathbb{P}^1 \setminus 3\,\mathrm{pt}) = -1$$

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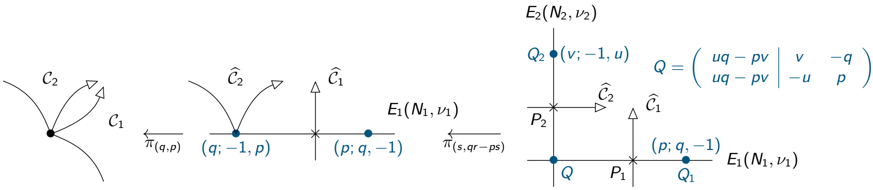

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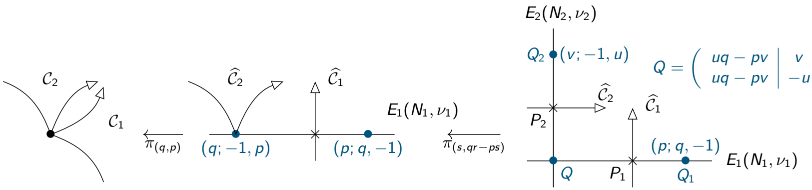

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Let $g(x,y,z) = x^p + y^q + z^r$, $p,q,r \in \mathbb{N}$ pairwise coprimes, $\omega = (qr, pr, pq)$

$D = V(g) \subset \mathbb{C}^3$, has an isolated singularity at the origin.

$$\begin{tikzcd} E \subset \widehat{\mathbb{C}}^3_\omega \arrow[d, two heads, "\pi_\omega"'] \\ \{0\} \subset \mathbb{C}^3 \end{tikzcd}$$

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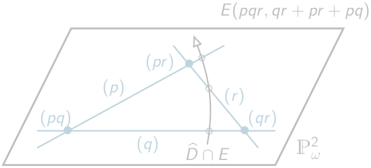

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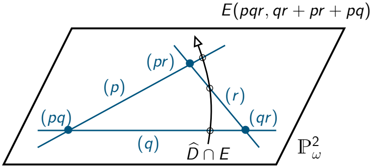

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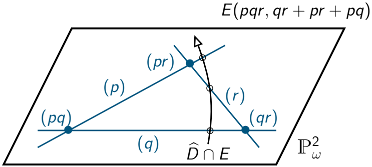

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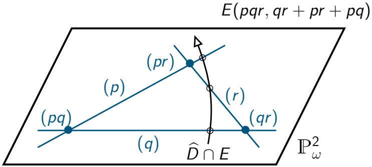

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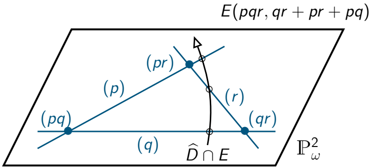

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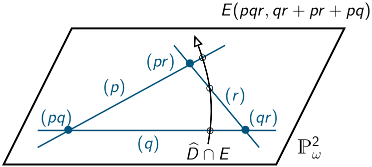

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Example: Brieskorn-Pham surface singularity

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$D = V(g) \subset \mathbb{C}^3$, has an isolated singularity at the origin.

$$\begin{array}{c} E \subset \widehat{\mathbb{C}}^3_\omega \simeq U_1 \cup U_2 \cup U_3, \quad U_i = \mathbb{C}^3/G_i \\ \Big\downarrow \pi_\omega \\ \{0\} \subset \mathbb{C}^3 \end{array}$$

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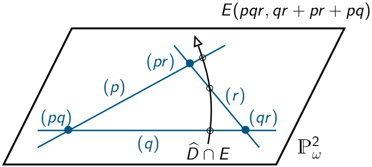

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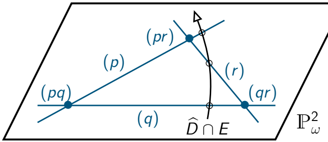

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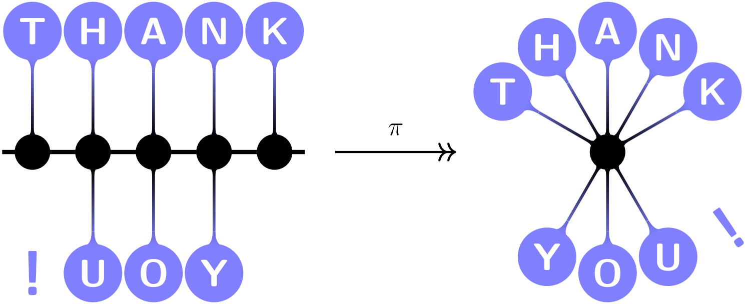
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